

# Asymmetric Quantum Codes: New Codes from Old

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**Abstract**—In this paper we extend to asymmetric quantum error-correcting codes (AQECC) the construction methods, namely: puncturing, extending, expanding, direct sum and the  $(\mathbf{u}|\mathbf{u} + \mathbf{v})$  construction. By applying these methods, several families of asymmetric quantum codes can be constructed. Consequently, as an example of application of quantum code expansion developed here, new families of asymmetric quantum codes derived from generalized Reed-Muller (GRM) codes, quadratic residue (QR), Bose-Chaudhuri-Hocquenghem (BCH), character codes and affine-invariant codes are constructed.

## I. INTRODUCTION

To make reliable the transmission or storage of quantum information against noise caused by the environment there exist many works available in the literature dealing with constructions of efficient quantum error-correcting codes (QECC) over unbiased quantum channels [3–5, 11, 16, 17, 20, 26, 30]. Recently, these constructions have been extended to asymmetric quantum channels in a natural way [1, 2, 7–10, 15, 18, 19, 21, 27, 28, 31, 32].

Asymmetric quantum error-correcting codes (AQECC) are quantum codes defined over quantum channels where qudit-flip errors and phase-shift errors may have different probabilities. Steane [29] was the first author who introduced the notion of asymmetric quantum errors. As usual, the parameters  $[[n, k, d_z/d_x]]_q$  denote an asymmetric quantum code, where  $d_z$  is the minimum distance corresponding to phase-shift errors and  $d_x$  is the minimum distance corresponding to qudit-flip errors. The combined amplitude damping and dephasing channel (specific to binary systems; see [27]) is an example for a quantum channel that satisfies  $d_z > d_x$ , i. e., the probability of occurrence of phase-shift errors is greater than the probability of occurrence of qudit-flip errors.

Let us give a brief summary of the papers available in the literature dealing with AQECC. In [7], the authors explored the asymmetry between qubit-flip and phase-shift errors to perform an optimization when compared to QECC. In [15] the authors utilize BCH codes to correct qubit-flip errors and LDPC codes to correct more frequently phase-shift errors. In [31] the authors consider the investigation of AQECC via code conversion. In the papers [1, 18], families of AQECC derived from BCH codes were constructed. Asymmetric stabilizer codes derived from LDPC codes were constructed in [27], and in [28], the same authors have constructed several families of both binary and nonbinary AQECC as well as to derive bounds such as the (quantum) Singleton and the linear programming bound to AQECC. In [2], both AQECC (derived from cyclic codes) and subsystem codes were investigated. In [32], the

construction of nonadditive AQECC as well as constructions of asymptotically good AQECC derived from algebraic-geometry codes were presented. In [9], the Calderbank-Shor-Steane (CSS) construction [5, 16, 24] was extended to include codes endowed with the Hermitian and also trace Hermitian inner product. In [8], asymmetric quantum MDS codes derived from generalized Reed-Solomon (GRS) codes were constructed. More recently, in [19, 21], constructions of families of AQECC by expanding GRS codes and by applying product codes, respectively, were presented.

In this paper we extend to asymmetric quantum error-correcting codes (AQECC) the construction methods, namely: puncturing, extending, expanding, direct sum and the  $(\mathbf{u}|\mathbf{u} + \mathbf{v})$  construction. An interesting fact pointed out by the referee is that the results presented in the first version of this paper (constructions of asymmetric quantum codes derived from classical linear codes endowed with the Euclidean as well as with the Hermitian inner product) also hold in a more general setting, i. e., constructions of asymmetric quantum codes derived from additive codes (see [5, 16], where in [5] a general theory of quantum codes over  $GF(4)$  was developed, and in [16] a generalization to nonbinary alphabets was presented). Because of this fact, we keep the original constructions of AQECC derived from linear codes and we also add more results with respect to constructions of AQECC derived from additive codes. More specifically, concerning the techniques of extending and the  $(\mathbf{u}|\mathbf{u} + \mathbf{v})$  construction, the arguments shown in this paper to AQECC codes derived from linear codes are similar to the ones derived from additive codes. The techniques of puncturing, expanding and direct sum will be shown in two different ways (each of them), i. e., AQECC derived from linear and additive codes. We keep both styles of constructions (additive/linear) in this paper because although the first (additive) is more general, we utilize different tools to show the results for the linear case, and these tools can be applied in future works.

The paper is organized as follows. In Section II we fix the notation. In Section III we recall the concepts and definitions of AQECC and error operators. Section IV is devoted to establish the construction methods. More precisely, we show how to construct new AQECC by means of the techniques of puncturing, extending, expanding, direct sum and the  $(\mathbf{u}|\mathbf{u} + \mathbf{v})$  construction. In Section V, we utilize the quantum code expansion developed in Section IV applied to (classical) generalized Reed-Muller (GRM) codes, quadratic residue, character codes, BCH and affine-invariant codes in order to construct several new families of AQECC. Finally, in Section VI, we discuss the contributions presented in this paper.

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## II. NOTATION

Throughout this paper,  $p$  denotes a prime number,  $q$  denotes a prime power,  $\mathbb{F}_q$  is a finite field with  $q$  elements,  $\alpha \in \mathbb{F}_{q^m}$  is a primitive  $n$  root of unity. The (Hamming) distance of two vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{F}_q^n$  is the number of coordinates in which  $\mathbf{v}$  and  $\mathbf{w}$  differ. The (Hamming) weight of a vector  $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{F}_q^n$  is the number of nonzero coordinates of  $\mathbf{v}$ . The trace map  $\text{tr}_{q^m/q} : \mathbb{F}_{q^m} \rightarrow \mathbb{F}_q$  is defined as  $\text{tr}_{q^m/q}(a) := \sum_{i=0}^{m-1} a^{q^i}$ . We denote  $H \leq G$  to mean that  $H$  is a subgroup of a group  $G$ ; the center of  $G$  is denoted by  $Z(G)$ . If  $S \leq G$  then we denote by  $C_G(S)$  the centralizer of  $S$  in  $G$ ;  $\langle S \rangle$  denotes the subgroup generated by  $S$  and the center  $Z(G)$ .

As usual,  $[n, k, d]_q$  denotes the parameters of a classical linear code  $C$  over  $\mathbb{F}_q$ , of length  $n$ , dimension  $k$  and minimum distance  $d$ . We denote by  $\text{wt}(C)$  the minimum weight of  $C$ , and by  $d(C)$  the minimum distance of  $C$ . Sometimes we have abused the notation by writing  $C = [n, k, d]_q$ . If  $C$  is an  $[n, k, d]_q$  code then its Euclidean dual is defined as  $C^\perp = \{\mathbf{y} \in \mathbb{F}_q^n \mid \mathbf{y} \cdot \mathbf{x} = 0, \forall \mathbf{x} \in C\}$ ; in the case that  $C$  is an  $[n, k, d]_{q^2}$  code, then its Hermitian dual is defined by  $C^{\perp_h} = \{\mathbf{y} \in \mathbb{F}_{q^2}^n \mid \mathbf{y}^q \cdot \mathbf{x} = 0, \forall \mathbf{x} \in C\}$ , where  $\mathbf{y}^q = (y_1^q, \dots, y_n^q)$  denotes the conjugate of the vector  $\mathbf{y} = (y_1, \dots, y_n)$ . If  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\mathbf{b} = (b_1, \dots, b_n)$  are two vectors in  $\mathbb{F}_q^n$  then the symplectic weight  $\text{swt}$  of the vector  $(\mathbf{a}|\mathbf{b}) \in \mathbb{F}_q^{2n}$  is defined by  $\text{swt}((\mathbf{a}|\mathbf{b})) = \#\{i : 1 \leq i \leq n \mid (a_i, b_i) \neq (0, 0)\}$ . The trace-symplectic form of two vectors  $(\mathbf{a}|\mathbf{b}), (\mathbf{a}^*|\mathbf{b}^*) \in \mathbb{F}_q^{2n}$  is defined by  $\langle (\mathbf{a}|\mathbf{b}) | (\mathbf{a}^*|\mathbf{b}^*) \rangle_s = \text{tr}_{q/p}(\mathbf{b} \cdot \mathbf{a}^* - \mathbf{b}^* \cdot \mathbf{a})$ . If  $C \leq \mathbb{F}_q^{2n}$  is an additive code then  $\text{swt}(C)$  denotes the symplectic weight of  $C$  and  $C^{\perp_s}$  denotes the trace-symplectic dual of  $C$ . Similarly, if  $C \leq \mathbb{F}_{q^2}^n$  is an additive code then  $C^{\perp_a}$  denotes the trace-alternating dual of  $C$ , where the trace-alternating form of two vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{F}_{q^2}^n$  is defined as  $\langle \mathbf{v} | \mathbf{w} \rangle_a = \text{tr}_{q/p} \left( \frac{\mathbf{v} \cdot \mathbf{w}^q - \mathbf{v}^q \cdot \mathbf{w}}{\beta^{2q} - \beta^2} \right)$ , where  $(\beta, \beta^q)$  is a normal basis of  $\mathbb{F}_{q^2}$  over  $\mathbb{F}_q$ .

## III. ERROR GROUPS AND ASYMMETRIC CODES

In this section we recall some basic concepts on quantum error operators [5, 16, 28] and asymmetric quantum codes.

Let  $\mathcal{H}$  be the Hilbert space  $\mathcal{H} = \mathbb{C}^{q^n} = \mathbb{C}^q \otimes \dots \otimes \mathbb{C}^q$ . Let  $|x\rangle$  be the vectors of an orthonormal basis of  $\mathbb{C}^q$ , where the labels  $x$  are elements of  $\mathbb{F}_q$ . Consider  $a, b \in \mathbb{F}_q$ ; the unitary operators  $X(a)$  and  $Z(b)$  on  $\mathbb{C}^q$  are defined by  $X(a)|x\rangle = |x+a\rangle$  and  $Z(b)|x\rangle = w^{\text{tr}_{q/p}(bx)}|x\rangle$ , respectively, where  $w = \exp(2\pi i/p)$  is a  $p$ th root of unity.

Consider that  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{F}_q^n$  and  $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{F}_q^n$ . Denote by  $X(\mathbf{a}) = X(a_1) \otimes \dots \otimes X(a_n)$  and  $Z(\mathbf{b}) = Z(b_1) \otimes \dots \otimes Z(b_n)$  the tensor products of  $n$  error operators. The set  $\mathbf{E}_n = \{X(\mathbf{a})Z(\mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n\}$  is an error basis on the complex vector space  $\mathbb{C}^{q^n}$  and the set  $\mathbf{G}_n = \{w^c X(\mathbf{a})Z(\mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathbb{F}_q^n, c \in \mathbb{F}_p\}$  is the error group associated with  $\mathbf{E}_n$ . For a quantum error  $e = w^c X(\mathbf{a})Z(\mathbf{b}) \in \mathbf{G}_n$  the  $X$ -weight is given by  $\text{wt}_X(e) = \#\{i : 1 \leq i \leq n \mid a_i \neq 0\}$ ; the  $Z$ -weight is defined as  $\text{wt}_Z(e) = \#\{i : 1 \leq i \leq n \mid b_i \neq 0\}$  and the symplectic (or quantum) weight  $\text{swt}(e) = \#\{i :$

$1 \leq i \leq n \mid (a_i, b_i) \neq (0, 0)\}$ . An AQECC with parameters  $((n, K, d_z/d_x))_q$  is an  $K$ -dimensional subspace of the Hilbert space  $\mathbb{C}^{q^n}$  and corrects all qudit-flip errors up to  $\lfloor \frac{d_x-1}{2} \rfloor$  and all phase-shift errors up to  $\lfloor \frac{d_z-1}{2} \rfloor$ . An  $((n, q^k, d_z/d_x))_q$  code is denoted by  $[[n, k, d_z/d_x]]_q$ .

Let us recall the well-known CSS construction:

**Lemma 3.1:** [5, 16, 24](CSS construction) Let  $C_1$  and  $C_2$  denote two classical linear codes with parameters  $[n, k_1, d_1]_q$  and  $[n, k_2, d_2]_q$ , respectively. Assume that  $C_2 \subset C_1$ . Then there exists an AQECC with parameters  $[[n, K = k_1 - k_2, d_z/d_x]]_q$ , where  $d_x = \text{wt}(C_2^\perp \setminus C_1^\perp)$  and  $d_z = \text{wt}(C_1 \setminus C_2)$ . The resulting code is said pure if, in the above construction,  $d_x = d(C_2^\perp)$  and  $d_z = d(C_1)$ .

Since the Euclidean dual of a code  $C$  and its Hermitian dual are isomorphic under Galois conjugation that preserves Hamming metric, a similar result can be derived if one considers in Lemma 3.1 the Hermitian inner product instead of considering the Euclidean inner product and we shall call the mentioned construction by CSS-type construction. Recently, the CSS construction was extended to include additive codes [9, Theorem 4.5].

The following result shown in [16] will be utilized in this paper:

**Theorem 3.2:** [16, Theorem 13] An  $((n, K, d))_q$  stabilizer code exists if and only if there exists an additive code  $C \leq \mathbb{F}_q^{2n}$  of size  $|C| = q^n/K$  such that  $C \leq C^{\perp_s}$  and  $\text{swt}(C^{\perp_s} \setminus C) = d$  if  $K > 1$  (and  $\text{swt}(C^{\perp_s}) = d$  if  $K = 1$ ).

## IV. CONSTRUCTION METHODS

This section is devoted to construct new AQECC from old ones. More precisely, we show how to obtain new codes by extending, puncturing, expanding, applying the direct sum and, finally, by using the  $(\mathbf{u}|\mathbf{u} + \mathbf{v})$  construction. In other words, we extend to AQECC all those methods valid to QECC.

### A. Code Expansion

Let us recall the concept of dual basis [22]. Given a basis  $\beta = \{b_1, b_2, \dots, b_m\}$  of  $\mathbb{F}_{q^m}$  over  $\mathbb{F}_q$ , a dual basis of  $\beta$  is given by  $\beta^\perp = \{b_1^*, b_2^*, \dots, b_m^*\}$ , with  $\text{tr}_{q^m/q}(b_i b_j^*) = \delta_{ij}$ , for all  $i, j \in \{1, \dots, m\}$ . A self-dual basis  $\beta$  is a basis satisfying  $\beta = \beta^\perp$ . If  $C$  is an  $[n, k, d_1]_{q^m}$  code and  $\beta = \{b_1, b_2, \dots, b_m\}$  is a basis of  $\mathbb{F}_{q^m}$  over  $\mathbb{F}_q$ , then the  $q$ -ary expansion  $\beta(C)$  of  $C$  with respect to  $\beta$  is an  $[mn, mk, d_2 \geq d_1]_q$  code given by  $\beta(C) := \{(c_{ij})_{i,j} \in \mathbb{F}_q^{mn} \mid \mathbf{c} = (\sum_j c_{ij} b_j)_i \in C\}$ .

**Lemma 4.1:** [3, 12, 19] Let  $C = [n, k, d]_{q^m}$  be a linear code over  $\mathbb{F}_{q^m}$ , where  $q$  is a prime power. Let  $C^\perp$  be the dual of the code  $C$ . Then the dual code of the  $q$ -ary expansion  $\beta(C)$  of code  $C$  with respect to the basis  $\beta$  is the  $q$ -ary expansion  $\beta^\perp(C^\perp)$  of the dual code  $C^\perp$  with respect to  $\beta^\perp$ .

Theorem 4.2 presents a method to construct AQECC by expanding linear codes:

**Theorem 4.2:** Let  $q$  be a prime power. Assume that there exists an AQECC with parameters  $[[n, k, d_z/d_x]]_{q^m}$ , derived from linear codes  $C_1 = [n, k_1, d_1]_{q^m}$  and  $C_2 = [n, k_2, d_2]_{q^m}$ , respectively. Then there exists an AQECC with parameters

$[[mn, mk, d_z^*/d_x^*]]_q$ , where  $k = k_1 - k_2$ ,  $d_z^* \geq d_1$  and  $d_x^* \geq d_2^\perp$ , where  $d_2^\perp$  denotes the minimum distance of the dual code  $C_2^\perp$ .

*Proof:* The proof presented here utilizes the same idea and generalizes the proof of [19, Theorem 1] to all linear codes. We begin by observing that  $[\beta(C)]^\perp = \beta^\perp(C^\perp)$ . Let  $C_1 = [n, k_1, d_1]_{q^m}$  and  $C_2 = [n, k_2, d_2]_{q^m}$  be two codes such that  $C_2 \subset C_1$ . Let  $\beta$  be any basis of  $\mathbb{F}_{q^m}$  over  $\mathbb{F}_q$  and  $\beta^\perp$  its dual basis. Consider the expansions  $\beta(C_1)$  of  $C_1$  and  $\beta(C_2)$  of  $C_2$  with respect to  $\beta$ . Then the inclusion  $\beta(C_2) \subset \beta(C_1)$  holds. The codes  $\beta(C_1)$ ,  $\beta(C_2)$  and  $[\beta(C_2)]^\perp$  are linear. Further,  $\beta(C_1) = [mn, mk_1, D_1 \geq d_1]_q$  and  $\beta(C_2) = [mn, mk_2, D_2 \geq d_2]_q$ , respectively. Since  $C_2^\perp$  has minimum distance  $d_2^\perp$ , then  $\beta^\perp(C_2^\perp)$  has minimum distance greater than or equal to  $d_2^\perp$  ( $\beta^\perp$  is a basis of  $\mathbb{F}_{q^m}$  over  $\mathbb{F}_q$ ). From Lemma 4.1 the equality  $[\beta(C_2)]^\perp = \beta^\perp(C_2^\perp)$  holds, hence  $[\beta(C_2)]^\perp$  also has minimum distance greater than or equal to  $d_2^\perp$ . Applying the CSS construction to  $\beta(C_1)$ ,  $\beta(C_2)$  and  $[\beta(C_2)]^\perp$ , one obtains an  $[[mn, m(k_1 - k_2), d_z^*/d_x^*]]_q$  asymmetric quantum code, where  $d_z^* \geq d_1$  and  $d_x^* \geq d_2^\perp$ . ■

More generally one has the following result:

*Theorem 4.3:* Let  $q = p^t$  be a prime power. If there exists an  $((n, K, d_z/d_x))_{q^m}$  stabilizer code then there exists an  $((nm, K, d_z^*/d_x^*))_{q^m}$  stabilizer code, where  $d_z^* \geq d_z$  and  $d_x^* \geq d_x$ .

*Proof:* If  $a$  is an element of  $\mathbb{F}_{q^m}$ , we can expand  $a$  with respect to a given basis  $B = \{\beta_1, \dots, \beta_m\}$  of  $\mathbb{F}_{q^m}$  over  $\mathbb{F}_q$  and put the coordinates of  $a$  in the vector form  $c_B(a) = (a_1, \dots, a_m) \in \mathbb{F}_q^m$ . Consider the non-degenerate symmetric form  $\text{tr}_{q^m/q}(ab)$  on the vector space  $\mathbb{F}_{q^m}$  (over  $\mathbb{F}_q$ ). Assume that  $\varphi_B$  is the  $\mathbb{F}_p$ -vector space isomorphism from  $\mathbb{F}_{q^m}^{2n}$  to  $\mathbb{F}_q^{2nm}$  given (in the proof of [16, Lemma 76]) by  $\varphi_B((\mathbf{u}|\mathbf{v})) = ((c_B(u_1), \dots, c_B(u_n)) | (Mc_B(v_1), \dots, Mc_B(v_n)))$ , where  $\mathbf{u}, \mathbf{v} \in \mathbb{F}_{q^m}^n$  are given by  $\mathbf{u} = (u_1, \dots, u_n)$  and  $\mathbf{v} = (v_1, \dots, v_n)$ ,  $M = (\text{tr}_{q^m/q}(\beta_i \beta_j))_{1 \leq i, j \leq n}$  denotes the Gram matrix and  $\text{tr}_{q^m/q}(ab) = c_B(a)^t Mc_B(b)$  for all  $a, b \in \mathbb{F}_{q^m}$ . Note that the inner product considered here is the usual (Euclidean) inner product of  $\mathbb{F}_q$ .

Assume that an  $((n, K, d_z/d_x))_{q^m}$  stabilizer code exists. From [16, Theorem 13], there exists an additive code  $C \leq \mathbb{F}_{q^m}^{2n}$  of size  $|C| = q^{mn}/K$  such that  $C \leq C^{\perp_s}$ ,  $\text{wt}_X(C^{\perp_s} \setminus C) = d_x$  if  $K > 1$  (and  $\text{wt}_X(C^{\perp_s}) = d_x$  if  $K = 1$ ) and  $\text{wt}_Z(C^{\perp_s} \setminus C) = d_z$  if  $K > 1$  (and  $\text{wt}_Z(C^{\perp_s}) = d_z$  if  $K = 1$ ). We know that  $\varphi_B$  preserves trace-symplectic orthogonality, i. e., the code  $\varphi_B(C)$  satisfies  $\varphi_B(C) \leq [\varphi_B(C)]^{\perp_s}$ . If  $(\mathbf{u}|\mathbf{v}) \in \mathbb{F}_{q^m}^{2n}$  and  $u_i \neq 0$  (resp.  $v_j \neq 0$ ) for some  $i \in \{1, \dots, n\}$  (resp.  $j \in \{1, \dots, n\}$ ), then at least one coordinate of the corresponding vector  $c_B(u_i)$  (resp.  $Mc_B(v_j)$ ) is nonzero. Thus  $\text{wt}_X([\varphi_B(C)]^{\perp_s} \setminus \varphi_B(C)) \geq d_x$  if  $K > 1$  (and  $\text{wt}_X([\varphi_B(C)]^{\perp_s}) \geq d_x$  if  $K = 1$ ) and  $\text{wt}_Z([\varphi_B(C)]^{\perp_s} \setminus \varphi_B(C)) \geq d_z$  if  $K > 1$  (and  $\text{wt}_Z([\varphi_B(C)]^{\perp_s}) \geq d_z$  if  $K = 1$ ). Because the alphabet considered now is  $\mathbb{F}_q$ , then there exists an  $((nm, K, d_z^*/d_x^*))_q$  stabilizer code, where  $d_z^* \geq d_z$  and  $d_x^* \geq d_x$ . ■

## B. Direct Sum Codes

Let us recall the direct sum of codes. Assume that  $C_1 = [n_1, k_1, d_1]_q$  and  $C_2 = [n_2, k_2, d_2]_q$  are two linear codes. Then the direct sum code  $C_1 \oplus C_2$  is the linear code given by  $C_1 \oplus C_2 = \{(\mathbf{c}_1, \mathbf{c}_2) | \mathbf{c}_1 \in C_1, \mathbf{c}_2 \in C_2\}$  and has parameters  $[n_1 + n_2, k_1 + k_2, \min\{d_1, d_2\}]_q$ .

*Theorem 4.4:* Let  $q$  be a prime power. Assume there exists an AQECC with parameters  $[[n, k, d_z/d_x]]_q$  derived from linear codes  $C_1 = [n, k_1, d_1]_q$  and  $C_2 = [n, k_2, d_2]_q$  with  $C_2 \subset C_1$ . Suppose also there exists an  $[[n^*, k^*, d_z^*/d_x^*]]_q$  AQECC derived from classical linear codes  $C_3 = [n^*, k_3, d_3]_q$  and  $C_4 = [n, k_4, d_4]_q$  with  $C_4 \subset C_3$ . Then there exists an  $[[n + n^*, k + k^*, d_z^\circ/d_x^\circ]]_q = [[n + n^*, (k_1 + k_3) - (k_2 + k_4), d_z^\circ/d_x^\circ]]_q$  AQECC, where  $d_z^\circ \geq \min\{d_1, d_3\}$ ,  $d_x^\circ \geq \min\{d_2^\perp, d_4^\perp\}$  and  $d_2^\perp, d_4^\perp$  are the minimum distances of the dual codes  $C_2^\perp$  and  $C_4^\perp$ , respectively.

*Proof:* Consider the direct sum codes  $C_1 \oplus C_3 = [n + n^*, k_1 + k_3, \min\{d_1, d_3\}]_q$  and  $C_2 \oplus C_4 = [n + n^*, k_2 + k_4, \min\{d_2, d_4\}]_q$ . Since the inclusions  $C_2 \subset C_1$  and  $C_4 \subset C_3$  hold it follows that the inclusion  $C_2 \oplus C_4 \subset C_1 \oplus C_3$  also holds. We know that a parity check matrix of the code  $(C_2 \oplus C_4)^\perp$  is given by  $G_2 \oplus G_4 = \begin{bmatrix} G_2 & 0 \\ 0 & G_4 \end{bmatrix}$ . Thus the minimum distance of  $(C_2 \oplus C_4)^\perp$  is equal to  $\min\{d_2^\perp, d_4^\perp\}$ . Therefore, applying the CSS construction to the codes  $C_1 \oplus C_3$ ,  $C_2 \oplus C_4$  and  $(C_2 \oplus C_4)^\perp$  one obtains an  $[[n + n^*, (k_1 + k_3) - (k_2 + k_4), d_z^\circ/d_x^\circ]]_q$  AQECC, where  $d_z^\circ \geq \min\{d_1, d_3\}$  and  $d_x^\circ \geq \min\{d_2^\perp, d_4^\perp\}$ . ■

The previous result also holds in a more general setting:

*Theorem 4.5:* Assume that there exist two stabilizer codes with parameters  $((n_1, K_1, d_z^{(1)}/d_x^{(1)}))_q$  and  $((n_2, K_2, d_z^{(2)}/d_x^{(2)}))_q$ . Then there exists an  $((n_1 + n_2, K_1 K_2, d_z^*/d_x^*))_q$ , where  $d_z^* = \min\{d_z^{(1)}, d_z^{(2)}\}$  and  $d_x^* = \min\{d_x^{(1)}, d_x^{(2)}\}$ .

*Proof:* The proof follows the same line of [16, Lemma 73]. We only show the result in the case of  $X$ -weight (the proof for  $Z$ -weight is similar). Note that if  $((n_1, K_1, d_z^{(1)}/d_x^{(1)}))_q$  and  $((n_2, K_2, d_z^{(2)}/d_x^{(2)}))_q$  are stabilizer codes with orthogonal projectors  $P_1$  and  $P_2$  respectively, and stabilizer  $S_1$  and  $S_2$  respectively, then  $P_1 \otimes P_2$  is an orthogonal projector onto a  $K_1 K_2$ -dimensional subspace  $Q^\oplus$  of  $\mathbb{C}^{q^{(n_1+n_2)}}$ , and the stabilizer of  $Q^\oplus$  is given by  $S^\oplus = \{E_1 \otimes E_2 | E_1 \in S_1, E_2 \in S_2\}$ . Assume that  $F_1 \otimes F_2 \in \mathbf{G}_{n_1} \otimes \mathbf{G}_{n_2}$  is not detectable; hence  $F_1 \in C_{\mathbf{G}_{n_1}}(S_1)$  and  $F_2 \in C_{\mathbf{G}_{n_2}}(S_2)$ . Moreover, either  $F_1 \notin S_1 Z(\mathbf{G}_{n_1})$  or  $F_2 \notin S_2 Z(\mathbf{G}_{n_2})$ , otherwise  $F_1 \otimes F_2$  would be detectable. Thus, from [16, Lemma 11], either  $F_1$  or  $F_2$  is not detectable, so  $\text{wt}_X(F_1 \otimes F_2)$  is at least  $\min\{d_x^{(1)}, d_x^{(2)}\}$ , and the result follows. ■

## C. Puncturing Codes

The technique of puncturing codes is well-known in the literature as in the classical [14, 23] as well as in the quantum case [5, 16, 26]. In this section we show how to construct AQECC by puncturing classical codes.

Let  $C$  be an  $[n, k, d]_q$  code. Then we denote by  $C^{P_i}$  the punctured code in the coordinate  $i$ . Recall that the dual of

a punctured code is a shortened code. Now we are ready to show the main result of this subsection:

**Theorem 4.6:** Assume that there exists an  $[[n, k, d_z/d_x]]_q$  stabilizer code derived from two linear codes  $C_1 = [n, k_1, d_1]_q$  and  $C_2 = [n, k_2, d_2]_q$  with  $C_2 \subset C_1$ ,  $n \geq 2$ ,  $k = k_1 - k_2$ ,  $d_z \geq d_1$  and  $d_x \geq d_2^\perp$ , where  $d_2^\perp$  is the minimum distance of the dual code  $C_2^\perp$ . Suppose also that  $d_1 \geq 2$ ,  $d_2^\perp \geq 2$  and  $C_2^\perp$  contains at least a nonzero codeword with  $i$ th coordinate zero. Then the following hold:

- (i) If  $C_1$  has a minimum weight codeword with a nonzero  $i$ th coordinate then there exists an  $[[n-1, k, d_z^i/d_x^i]]_q$  AQECC, where  $k = k_1 - k_2$ ,  $d_z^i \geq d_1 - 1$  and  $d_x^i \geq d_2^\perp$ ;
- (ii) If  $C_1$  has no minimum weight codeword with a nonzero  $i$ th coordinate, then there exists an  $[[n-1, k, d_z^i/d_x^i]]_q$  AQECC, where  $k = k_1 - k_2$ ,  $d_z^i \geq d_1$  and  $d_x^i \geq d_2^\perp \geq 2$ .

*Proof:* We only prove item (ii) since the proof of (i) is similar to this one. Consider the punctured codes  $C_1^{P_i}$  and  $C_2^{P_i}$ . Since the inclusion  $C_2 \subset C_1$  holds it follows that  $C_2^{P_i} \subset C_1^{P_i}$ . Since from hypothesis one has  $d_1 > 1$  then it follows that  $d_2 > 1$  because  $C_2 \subset C_1$ ; again from the hypothesis  $C_1$  has no minimum weight codeword with a nonzero  $i$ th coordinate. Thus, by Theorem [14, Theorem 1.5.1], the punctured codes  $C_1^{P_i}$  and  $C_2^{P_i}$  have parameters  $[n-1, k_1, d_1]_q$  and  $[n-1, k_2, d_2^i]_q$ , respectively, where  $d_2^i = d_2$  or  $d_2^i = d_2 - 1$ .

We need to compute the minimum distance of the code  $[C_2^{P_i}]^\perp$  in order to apply the CSS construction. To do this consider the code  $[C_2^{P_i}]^\perp$ . Since  $C_2^\perp$  contains at least a nonzero codeword whose  $i$ th coordinate is equal to zero then  $C_2^\perp$  has a subcode  $C_2^\perp(\{i\}) \neq \{0\}$  and, consequently, the minimum distance  $d_{(C_2^\perp)_i}$  of  $C_2^\perp(\{i\})$  satisfies  $d_{(C_2^\perp)_i} \geq d_2^\perp$ , where  $d_2^\perp > 1$ . Since  $d_{(C_2^\perp)_i} > 1$  and because (from definition) the code  $C_2^\perp(\{i\})$  has no minimum weight codeword with a nonzero  $i$ th coordinate, applying again Theorem [14, Theorem 1.5.1], it implies that the shortened code  $[C_2^\perp]_{S_i}$  has minimum distance equals  $d_{(C_2^\perp)_i}$ . From [14, Theorem 1.5.7] we know that  $[C_2^{P_i}]^\perp = [C_2^\perp]_{S_i}$ , so the code  $[C_2^{P_i}]^\perp$  has minimum distance  $d_{(C_2^\perp)_i}$ , where  $d_{(C_2^\perp)_i} \geq d_2^\perp$ . Therefore, applying the CSS construction to the codes  $C_1^{P_i}$ ,  $C_2^{P_i}$  and  $[C_2^{P_i}]^\perp$ , one can derive an  $[[n-1, k, d_z^i/d_x^i]]_q$  AQECC, where  $k = k_1 - k_2$ ,  $d_z^i \geq d_1$  and  $d_x^i \geq d_{(C_2^\perp)_i} \geq d_2^\perp \geq 2$ . ■

Following the lines adopted in [16] we can show a more general result:

**Theorem 4.7:** Assume that a pure  $[[n, k, d_z/d_x]]_q$  stabilizer code exists, with  $n \geq 2$  and  $d_x, d_z \geq 2$ . Then there exists a pure  $[[n-1, k, d_z^*/d_x^*]]_q$  stabilizer code, where  $d_z^* \geq d_z - 1$  and  $d_x^* \geq d_x - 1$ .

*Proof:* Assume that a pure  $[[n, k, d_z/d_x]]_q$  stabilizer code exists, with the corresponding minimum distance  $d$ . From [16, Corollary 72], there exists a pure  $[[n-1, k, d^* \geq d-1]]_q$  stabilizer code derived from an additive self-orthogonal (with respect to the trace-alternating form) code  $D^{\perp_a} \leq \mathbb{F}_q^{n-1}$  with  $\text{wt}(D^{\perp_a}) \geq d-1$ . Consider the vectors  $\mathbf{v}, \mathbf{w} \in \mathbb{F}_q^{2(n-1)}$  and let  $(\beta, \beta^q)$  be a normal basis of  $\mathbb{F}_q^2$  over  $\mathbb{F}_q$ . We know that the

bijective map  $\phi((\mathbf{v}|\mathbf{w})) = \beta\mathbf{v} + \beta^q\mathbf{w}$  from  $\mathbb{F}_q^{2(n-1)}$  onto  $\mathbb{F}_{q^2}^{n-1}$  is an isometry (symplectic/Hamming weights, resp.) (see also [16, Lemma 14]). Considering the inverse map  $\phi^{-1}$  and the corresponding additive code  $\phi^{-1}(D^{\perp_a}) \leq \mathbb{F}_q^{2(n-1)}$ , it follows that  $\phi^{-1}(D^{\perp_a})$  has minimum  $X$ -weight  $d_x^*$  at least  $d_x^* \geq d_x - 1$  and the minimum  $Z$ -weight  $d_z^*$  at least  $d_z^* \geq d_z - 1$ , and the proof is complete. ■

**Remark 4.8:** Note that the procedure adopted in Theorems 4.6 and 4.7 can be generalized by puncturing codes on two or more coordinates.

#### D. Code Extension

The technique of (classical) code extension [14, 23] was derived also in the quantum case [5, 16]. Here we extend to AQECC the referred technique.

Let  $C$  be an  $[n, k, d]_q$  linear code over  $\mathbb{F}_q$ . The extended code  $C^e$  is the linear code given by  $C^e = \{(x_1, \dots, x_n, x_{n+1}) \in \mathbb{F}_q^{n+1} \mid (x_1, \dots, x_n) \in C, x_1 + \dots + x_n + x_{n+1} = 0\}$ . The code  $C^e$  is linear and has parameters  $[n+1, k, d^e]_q$ , where  $d^e = d$  or  $d^e = d+1$ . Recall that a vector  $\mathbf{v} = (v_1, \dots, v_n) \in \mathbb{F}_q^n$  is called *even-like* if it satisfies the equality  $\sum_{i=1}^n v_i = 0$ , and *odd-like* otherwise. For

an  $[n, k, d]_q$  code  $C$  the minimum weight of the even-like codewords of  $C$  are called *minimum even-like weight* and denoted by  $d_{\text{even}}$  (or  $(d)_{\text{even}}$ ). Similarly, the minimum weight of the odd-like codewords of  $C$  are called *minimum odd-like weight* and denoted by  $d_{\text{odd}}$  (or  $(d)_{\text{odd}}$ ).

Let us now prove the main result of this subsection.

**Theorem 4.9:** Assume that there exists an  $[[n, k, d_z/d_x]]_q$  AQECC derived from codes  $C_1 = [n, k_1, d_1]_q$  and  $C_2 = [n, k_2, d_2]_q$ , where  $C_2 \subset C_1$ . Then the following hold:

- (a) If  $(d_1)_{\text{even}} \leq (d_1)_{\text{odd}}$ , then there exists an  $[[n+1, k, d_z^e/d_x^e]]_q$  AQECC, where  $d_z^e \geq d_1$  and  $d_x^e \geq (d_2^\perp)^\perp$ , where  $(d_2^\perp)^\perp$  is the minimum distance of the dual  $(C_2^\perp)^\perp$  of the extended code  $C_2^e$ ;
- (b) If  $(d_1)_{\text{odd}} < (d_1)_{\text{even}}$ , then there exists an  $[[n+1, k, d_z^e/d_x^e]]_q$  AQECC, where  $d_z^e \geq d_1 + 1$  and  $d_x^e \geq (d_2^\perp)^\perp$ .

*Proof:* We only show item (b), since (a) is similar. It is easy to see that the inclusion  $C_2^e \subset C_1^e$  holds. The parameters of the extended codes  $C_1^e$  and  $C_2^e$  are  $[n+1, k_1, d_1^e]_q$  and  $[n+1, k_2, d_2^e]_q$ , respectively, where  $d_1^e = d_1$  or  $d_1^e = d_1 + 1$ . Since  $(d_1)_{\text{odd}} < (d_1)_{\text{even}}$ , it follows from the remark shown in [14, pg. 15] that  $d_1^e = d_1 + 1$ . From hypothesis we know that  $k = k_1 - k_2$ , so the corresponding CSS code also has dimension  $k$ . Applying the CSS construction to the codes  $C_1^e$ ,  $C_2^e$  and  $(C_2^e)^\perp$ , one obtains an AQECC with parameters  $[[n+1, k, d_z^e/d_x^e]]_q$ , where  $d_z^e \geq d_1 + 1$  and  $d_x^e \geq (d_2^\perp)^\perp$ . ■

#### E. The $(\mathbf{u}|\mathbf{u} + \mathbf{v})$ Construction

The  $(\mathbf{u}|\mathbf{u} + \mathbf{v})$  construction [14, 23] is an interesting method for constructing new (classical) linear codes. Our intention is to apply this technique in order to generate a similar construction method for asymmetric quantum codes.

Let  $C_1$  and  $C_2$  be two linear codes of same length both over  $\mathbb{F}_q$  with parameters  $[n, k_1, d_1]_q$  and  $[n, k_2, d_2]_q$ , respectively. Then by applying the  $(\mathbf{u}|\mathbf{u}+\mathbf{v})$  construction one can generate a new code  $C' = \{(\mathbf{u}|\mathbf{u}+\mathbf{v})|\mathbf{u} \in C_1, \mathbf{v} \in C_2\}$  with parameters  $[2n, k_1 + k_2, \min\{2d_1, d_2\}]_q$ . To simplify the notation, we denote the code produced by applying the  $(\mathbf{u}|\mathbf{u}+\mathbf{v})$  construction to the codes  $C_1$  and  $C_2$  by  $(C_1|C_1 + C_2)$ .

Theorem 4.10 is the main result of this subsection:

**Theorem 4.10:** Assume that there exist two asymmetric stabilizer codes  $[[n, k^*, d_z^*/d_x^*]]_q$ , derived from codes  $C_1 = [n, k_1, d_1]_q$  and  $C_2 = [n, k_2, d_2]_q$  with  $C_2 \subset C_1$ , and  $[[n, k^\circ, d_z^\circ/d_x^\circ]]_q$ , derived from codes  $C_3 = [n, k_3, d_3]_q$  and  $C_4 = [n, k_4, d_4]_q$  with  $C_4 \subset C_3$ . Then there exists an  $[[2n, k^* + k^\circ, d_z/d_x]]_q$  AQECC, where  $d_z \geq \min\{2d_1, d_3\}$ ,  $d_x \geq \min\{2d_4^\perp, d_2^\perp\}$ , with  $d_z^* \geq d_1$ ,  $d_x^* \geq d_2^\perp$ ,  $d_z^\circ \geq d_3$  and  $d_x^\circ \geq d_4^\perp$ , where  $d_2^\perp$  and  $d_4^\perp$  are the minimum distances of the dual codes  $C_2^\perp$  and  $C_4^\perp$ , respectively.

*Proof:* Since the inclusions  $C_2 \subset C_1$  and  $C_4 \subset C_3$  hold it follows that the inclusion  $(C_2|C_2 + C_4) \subset (C_1|C_1 + C_3)$  also holds. We know that the codes  $(C_2|C_2 + C_4)$  and  $(C_1|C_1 + C_3)$  have parameters  $[2n, k_2 + k_4, \min\{2d_2, d_4\}]_q$  and  $[2n, k_1 + k_3, \min\{2d_1, d_3\}]_q$ , respectively. Let us compute the minimum distance of the dual code  $[(C_2|C_2 + C_4)]^\perp$ . We know that a generator matrix of  $[(C_2|C_2 + C_4)]^\perp$  is the matrix  $\begin{bmatrix} H_2 & 0 \\ -H_4 & H_4 \end{bmatrix}$ , where  $H_2$  and  $H_4$  are the parity check matrices of  $C_2$  and  $C_4$ , respectively. The codewords of  $[(C_2|C_2 + C_4)]^\perp$  are of the form  $\{(\mathbf{u} - \mathbf{v}, \mathbf{v})|\mathbf{u} \in C_2^\perp, \mathbf{v} \in C_4^\perp\}$ . Consider the codeword  $\mathbf{w} = (\mathbf{u} - \mathbf{v}, \mathbf{v})$ . If  $\mathbf{u} = 0$  then  $\mathbf{w} = (-\mathbf{v}, \mathbf{v})$ , so the minimum weight of  $[(C_2|C_2 + C_4)]^\perp$  is given by  $2d_4^\perp$ . On the other hand, if  $\mathbf{u} \neq 0$  then  $\text{wt}(\mathbf{w}) = \text{wt}(\mathbf{u} - \mathbf{v}) + \text{wt}(\mathbf{v}) = d(\mathbf{u}, \mathbf{v}) + d(\mathbf{v}, 0) \geq d(\mathbf{u}, 0) = \text{wt}(\mathbf{u})$ . Thus the minimum weight is given by  $d_2^\perp$  and, consequently, the minimum distance of  $[(C_2|C_2 + C_4)]^\perp$  is equal to  $\min\{2d_4^\perp, d_2^\perp\}$ . Applying the CSS construction to the codes  $(C_2|C_2 + C_4)$ ,  $(C_1|C_1 + C_3)$  and  $[(C_2|C_2 + C_4)]^\perp$ , one obtains an  $[[2n, (k_1 + k_3) - (k_2 + k_4), d_z/d_x]]_q = [[2n, k^* + k^\circ, d_z/d_x]]_q$  asymmetric stabilizer code, where  $d_z \geq \min\{2d_1, d_3\}$  and  $d_x \geq \min\{2d_4^\perp, d_2^\perp\}$ , as required.

As an alternative proof (suggested by the referee), we also can write the codewords of  $[(C_2|C_2 + C_4)]^\perp$  in the form  $\{(\mathbf{u} + \mathbf{v}, -\mathbf{v})|\mathbf{u} \in C_2^\perp, \mathbf{v} \in C_4^\perp\}$ , and because the Hamming weights of  $\mathbf{v}$  and  $-\mathbf{v}$  are the same, the latter code is equivalent to  $\{(\mathbf{u} + \mathbf{v}, \mathbf{v})|\mathbf{u} \in C_2^\perp, \mathbf{v} \in C_4^\perp\}$ , and the result follows. ■

## V. CODE CONSTRUCTIONS

In this section we utilize the construction methods developed in Section IV to obtain new families of AQECC. In order to shorten the length of this paper we only apply the quantum code expansion shown in Subsection IV-A of Section IV, although it is clear that all construction methods proposed in Section IV can also be applied. In Subsections V-A, V-B, V-C, V-D and V-E we construct AQECC derived from generalized Reed-Muller (GRM), character codes, BCH, quadratic residue (QR) and affine-invariant codes, respectively. In Subsection V-F, we construct a code table containing the param-

eters of known AQECC as well the parameters of the new codes.

**Remark 5.1:** It is important to observe that in all results presented in the following, we expand the codes defined over  $\mathbb{F}_q$  (where  $q = p^t$ ,  $t \geq 1$  and  $p$  prime) with respect to the prime field  $\mathbb{F}_p$ . However, the method also holds if one expands such a codes over any subfield of the field  $\mathbb{F}_q$ .

### A. Construction I- Generalized Reed-Muller Codes

The first family of AQECC derived from binary Reed-Muller (RM) codes were constructed in [28, Lemma 4.1]. In this subsection we present a construction of AQECC derived from generalized Reed-Muller (GRM) [23, 25].

The GRM code  $\mathcal{R}_q(\alpha, m)$  over  $\mathbb{F}_q$  of order  $\alpha$ ,  $0 \leq \alpha < q(m-1)$ , has parameters  $[q^m, k(\alpha), d(\alpha)]_q$ , where

$$k(\alpha) = \sum_{i=0}^m (-1)^i \binom{m}{i} \binom{m + \alpha - iq}{\alpha - iq} \quad (1)$$

and

$$d(\alpha) = (t+1)q^u, \quad (2)$$

where  $m(q-1) - \alpha = (q-1)u + t$  and  $0 \leq t < q-1$ . The dual of a GRM code  $\mathcal{R}_q(\alpha, m)$  is also a GRM code given by  $[\mathcal{R}_q(\alpha, m)]^\perp = \mathcal{R}_q(\alpha^\perp, m)$ , where  $\alpha^\perp = m(q-1) - 1 - \alpha$ .

We use the properties of the GRM codes in order to deriving new asymmetric quantum codes:

**Theorem 5.2:** Let  $0 \leq \alpha_1 \leq \alpha_2 < m(q-1)$  and assume that  $q = p^t$  is a prime power, where  $t \geq 1$ . Then there exists an  $p$ -ary asymmetric quantum GRM code with parameters  $[[tq^m, t[k(\alpha_2) - k(\alpha_1)], d_z/d_x]]_p$ , where  $d_z \geq d(\alpha_2)$ ,  $d_x \geq d(\alpha_1^\perp)$ ,  $k(\alpha_2)$  and  $k(\alpha_1)$  are given in Eq. (1),  $d(\alpha_2)$  is given in Eq. (2) and  $d(\alpha_1^\perp) = (a+1)q^b$ , where  $\alpha_1 + 1 = (q-1)b + a$  and  $0 \leq a \leq q-1$ .

*Proof:* First, note that since the inequality  $\alpha_1 \leq \alpha_2$  holds then the inclusion  $\mathcal{R}_q(\alpha_1, m) \subset \mathcal{R}_q(\alpha_2, m)$  also holds. The codes  $\beta(\mathcal{R}_q(\alpha_1, m))$  and  $\beta(\mathcal{R}_q(\alpha_2, m))$  have parameters  $[tq^m, tk(\alpha_1), d(\alpha_1)]_p$  and  $[tq^m, tk(\alpha_2), d(\alpha_2)]_p$ , respectively, where  $k(\alpha_1)$  and  $k(\alpha_2)$  are computed according to Eq. (1) and  $d(\alpha_1)$ ,  $d(\alpha_2)$  are computed by applying Eq. (2). We know that the parameter  $\alpha_1^\perp$  of the dual code  $[\mathcal{R}_q(\alpha_1, m)]^\perp = \mathcal{R}_q(\alpha_1^\perp, m)$  equals  $\alpha_1^\perp = m(q-1) - 1 - \alpha_1$ , so the minimum distance of  $[\mathcal{R}_q(\alpha_1, m)]^\perp$  is equal to  $d(\alpha_1^\perp) = (a+1)q^b$ , where  $\alpha_1 + 1 = (q-1)b + a$  and  $0 \leq a \leq q-1$ . Thus the code  $[\beta(\mathcal{R}_q(\alpha_1, m))]^\perp$  has minimum distance greater than or equal to  $d(\alpha_1^\perp)$ . Applying Theorem 4.2 one can get an  $[[tq^m, t[k(\alpha_2) - k(\alpha_1)], d_z/d_x]]_p$  asymmetric stabilizer code, where  $d_z \geq d(\alpha_2)$  and  $d_x \geq d(\alpha_1^\perp)$ . ■

### B. Construction II- Character Codes

The class of (classical) character codes were introduced by Ding *et al.* [6]. Let us consider the commutative group  $G = \mathbb{Z}_2^m$ ,  $m \geq 1$  and a finite field  $\mathbb{F}_q$  of odd characteristic. Recall that the code  $C_q(r, m) = C_X$ , where  $X \subset \mathbb{Z}_2^m$  consists of elements with Hamming weight greater than  $r$  has parameters  $[2^m, s_m(r), 2^{m-r}]_q$  (see [6, Theorem 6]), where

$s_m(r) = \sum_{i=0}^r \binom{m}{i}$ . The (Euclidean) dual code  $[C_q(r, m)]^\perp$  of  $C_q(r, m)$  is equivalent to  $C_q(m-r-1, m)$  (see [6, Theorem 8]) and consequently has parameters  $[2^m, s_m(m-r-1), 2^{r+1}]_q$ .

Next we utilize the code expansion applied to character codes to generate new AQECC, as established in the following theorem:

**Theorem 5.3:** If  $0 \leq r_1 < r_2 \leq m$  and  $q = p^t$  is a power of an odd prime  $p$ , where  $t \geq 1$ , then there exists an  $[[t2^m, t(k(r_2) - k(r_1)), d_z/d_x]]_p$  AQECC, where  $k(r) = \sum_{i=0}^r \binom{m}{i}$  and  $d_z \geq 2^{m-r_2}$  and  $d_x \geq 2^{r_1+1}$ .

*Proof:* It is easy to see that  $C_q(r_1, m) \subset C_q(r_2, m)$ . The dual code  $[C_q(r_1, m)]^\perp$  is equivalent to the code  $C_q(m-r_1-1, m)$ . Applying Theorem 4.2 one can get an  $[[t2^m, t(k(r_2) - k(r_1)), d_z/d_x]]_p$  AQECC, where  $t, k(r_1), k(r_2), d_x$  and  $d_z$  are specified in the hypothesis. ■

### C. Construction III - BCH Codes

In this subsection we construct more families of asymmetric stabilizer codes derived from Bose-Chaudhuri-Hocquenghem (BCH) codes [23]. The first families of AQECC derived from BCH codes were constructed by Aly [1, Theorem 8]. Recently, the parameters of these codes were improved for certain families of BCH codes [18].

Recall that a cyclic code of length  $n$  over  $\mathbb{F}_q$  is a BCH code with designed distance  $\delta$  if, for some integer  $b \geq 0$ , one has

$$g(x) = \text{l.c.m.}\{M^{(b)}(x), M^{(b+1)}(x), \dots, M^{(b+\delta-2)}(x)\},$$

i. e.,  $g(x)$  is the monic polynomial of smallest degree over  $\mathbb{F}_q$  having  $\alpha^b, \alpha^{b+1}, \dots, \alpha^{b+\delta-2}$  as zeros. The next result shows how to construct more AQECC by expanding (classical) BCH codes:

**Theorem 5.4:** Suppose that  $n = q^m - 1$ , where  $q = p^t$  is a power of an odd prime  $p$ ,  $t \geq 1$  and  $m \geq 3$  are integers an integer (if  $q = 3$ ,  $m \geq 4$ ). Then there exist quantum codes with parameters

- $[[tn, t(n-m(4q-5)-2), d_z \geq (2q+2)/d_x \geq 2q]]_p$ ;
- $[[tn, t(n-m(4q-c-5)-2), d_z \geq (2q+2)/d_x \geq (2q-c)]_p$ , where  $0 \leq c \leq q-2$ ;
- $[[tn, t(n-m(2c-l-4)-2), d_z \geq c/d_x \geq (c-l)]_p$ , where  $2 \leq c \leq q$  and  $0 \leq l \leq c-2$ ;
- $[[tn, t(n-m(2c-l-6)-2), d_z \geq c/d_x \geq (c-l)]_p$ , where  $q+2 < c \leq 2q$  and  $0 \leq l \leq c-q-3$ ;
- $[[tn, t(n-m(4q-l-5)-1), d_z \geq (2q+1)/d_x \geq (2q-l)]_p$ , where  $0 \leq l \leq q-2$ .

*Proof:* Consider the codes constructed in [18, Theorems 4 and 5 and Corollary 1]. These codes are derived from two distinct nested cyclic codes  $C_2 \subset C_1$ . Thus, applying Theorem 4.2 the result holds. ■

**Theorem 5.5:** Let  $q = p^t$  be a power of a prime  $p$ ,  $t \geq 1$ ,  $\gcd(q, n) = 1$  and  $\text{ord}_n(q) = m$ . Let  $C_1$  and  $C_2$  be two narrow-sense BCH codes of length  $q^{\lfloor m/2 \rfloor} < n \leq q^m - 1$  over  $\mathbb{F}_q$  with designed distances  $\delta_1$  and  $\delta_2$  in the range  $2 \leq \delta_1, \delta_2 \leq \delta_{\max} = \min\{\lfloor nq^{\lfloor m/2 \rfloor} / (q^m - 1) \rfloor, n\}$  and  $\delta_1 < \delta_2^\perp \leq \delta_2 <$

$\delta_1^\perp$ . Assume also that  $S_1 \cup \dots \cup S_{\delta_1-1} \neq S_1 \cup \dots \cup S_{\delta_2-1}$ , where  $S_i$  denotes a cyclotomic coset. Then there exists an AQECC with parameters  $[[tn, t(n-m(\delta_1-1)(1-1/q)) - m(\delta_2-1)(1-1/q)], d_z^*/d_x^*]_p$ , where  $d_z^* = wt(C_2 \setminus C_1^\perp) \geq \delta_2$  and  $d_x^* = wt(C_1 \setminus C_2^\perp) \geq \delta_1$ .

*Proof:* It suffices to apply Theorem 4.2 in those codes shown in [1, Theorem 8]. ■

**Remark 5.6:** Note that one can obtain more families of AQECC by applying Theorem 4.2 in the existing families shown in [17]. Moreover, expanding generalized Reed-Solomon (GRS) codes, one obtains [19, Theorem 7.1] as a particular case of Theorem 4.2.

### D. Construction IV- Quadratic Residue Codes

In this subsection we construct families of AQECC derived from quadratic residue (QR) codes [14, 23]. A family of quantum codes derived from classical QR codes was constructed in [16, Theorems 40 and 41].

Let  $p$  be an odd prime not dividing  $q$ , where  $q$  is a prime power that is a square modulo  $p$ . Let  $Q$  be the set of nonzero squares modulo  $p$  and  $C$  consisting of non-squares modulo  $p$ . The quadratic residue codes  $\mathcal{Q}$ ,  $\mathcal{Q}^\circ$ ,  $\mathcal{C}$  and  $\mathcal{C}^\circ$  are cyclic codes with generator polynomials  $q(x)$ ,  $(x-1)q(x)$ ,  $c(x)$ ,  $(x-1)c(x)$ , respectively, where  $q(x) = \prod_{r \in Q} (x - \alpha^r)$ ,  $c(x) =$

$\prod_{s \in C} (x - \alpha^s)$  have coefficients from  $\mathbb{F}_q$ , and  $\alpha$  is a primitive  $p$ th root of unity belonging to some extension field of  $\mathbb{F}_q$ . The codes  $\mathcal{Q}$  and  $\mathcal{C}$  have the same parameters  $[p, (p+1)/2, d_1]_q$ , where  $(d_1)^2 \geq p$ ; similarly, the codes  $\mathcal{Q}^\circ$  and  $\mathcal{C}^\circ$  also have the same parameters  $[p, (p-1)/2, d_2]_q$ , where  $(d_2)^2 \geq p$ .

Now we construct families of AQECC by expanding quadratic residue codes:

**Theorem 5.7:** Let  $p$  be a prime of the form  $p \equiv 1 \pmod{4}$ , and let  $q = p^t$  ( $t \geq 1$ ) be a power of a prime that is not divisible by  $p$ . If  $q$  is a quadratic residue modulo  $p$ , then there exists an  $[[tp, t, d_z/d_x]]_{p^*}$  asymmetric quantum code, where  $d_z$  and  $d_x$  satisfy  $d_z \geq \sqrt{p}$  and  $d_x \geq \sqrt{p}$ .

*Proof:* Consider the codes  $\mathcal{Q}$ ,  $\mathcal{Q}^\circ$  and  $\mathcal{C}$  given above. Since  $p = 4k+1$ , then it is well known that  $\mathcal{Q}^\circ = \mathcal{C}^\perp$ , so  $\mathcal{C}^\perp \subset \mathcal{Q}$ . The codes  $\mathcal{Q}$  and  $\mathcal{C}^\perp$  have parameters, respectively, given by  $[p, (p+1)/2, d_1]_q$ , with  $(d_1)^2 \geq p$  and  $[p, (p-1)/2, d_2]_q$ , where  $(d_2)^2 \geq p$ . Proceeding similarly as in the proof of Theorem 4.2 one can get an  $[[tp, t, d_z/d_x]]_{p^*}$  asymmetric quantum code, where  $d_z$  and  $d_x$  satisfy  $d_z \geq \sqrt{p}$  and  $d_x \geq \sqrt{p}$ . ■

**Theorem 5.8:** Let  $p$  be a prime of the form  $p \equiv 3 \pmod{4}$ , and let  $q = p^t$  ( $t \geq 1$ ) be a power of a prime that is not divisible by  $p$ . If  $q$  is a quadratic residue modulo  $p$ , then there exists an  $[[tp, t, d_z/d_x]]_{p^*}$  quantum code, where  $d_z \geq d$ ,  $d_x \geq d$  and  $d$  satisfies  $d^2 - d + 1 \geq p$ .

*Proof:* Since  $p = 4k-1$ , the dual  $\mathcal{Q}^\perp$  of  $\mathcal{Q}$  equals  $\mathcal{Q}^\perp = \mathcal{Q}^\circ$ , so  $\mathcal{Q}^\perp \subset \mathcal{Q}$ . The codes  $\mathcal{Q}$  and  $\mathcal{Q}^\perp$  have parameters  $[p, (p+1)/2, d]_q$  and  $[p, (p-1)/2, d^\circ \geq d]_q$ , respectively, and the minimum distance is bounded by  $d^2 - d + 1 \geq p$  (see for instance the proof of Theorem 40 in [16]). Applying

Theorem 4.3 one has an  $[[tp, t, d_z/d_x]]_{p^*}$  code, where  $d_z \geq d$ ,  $d_x \geq d$  and  $d^2 - d + 1 \geq p$ . ■

*Remark 5.9:* As observed by the referee, a refined statement can be made if one considers the code  $\mathcal{Q}^\circ$  instead of considering the code  $\mathcal{Q}$ , because  $d_{\mathcal{Q}^\circ} = d_{\mathcal{Q}} + 1$  (see [23, Chapter 16, Problem (2), p. 494]).

### E. Construction V- Affine-Invariant Codes

We assume that the reader is familiar with the class of (classical) affine-invariant codes. The structure and results on this class of codes can be found in [14].

Quantum affine-invariant codes were investigated in the literature [13]:

*Lemma 5.10:* [13, Lemma 22] Let  $C^e$  be an extended maximal affine-invariant code  $[[p^m, p^m - 1 - m/t, d]]_{p^t}$ , then if  $p > 3$  or  $m > 2$  or  $t \neq 1$ , we have  $(C^e)^\perp \subset C^e$ .

Applying Lemma 5.10 we can construct a family of AQECC derived from affine-invariant codes:

*Theorem 5.11:* Assume that  $q = p^t$ ,  $m$  is a positive integer and  $n = p^m - 1$ . If  $p > 3$  or  $m > 2$  or  $t \neq 1$  then there exists an AQECC com parameters  $[[tp^m, t(p^m - 2 - 2\frac{m}{t}), d_z/d_x]]_p$ , where  $d_z \geq d_a$ ,  $d_x \geq d_a$ , and  $d_a$  is the minimum distance of an extended maximal affine-invariant code.

*Proof:* Consider the dual containing extended maximal affine-invariant code  $C^e$  with parameters  $[[p^m, p^m - 1 - m/t, d]]$  given in Lemma 5.10, where  $p > 3$  (or  $m > 2$  or  $t \neq 1$ ). Applying Theorem 4.3 one obtains an  $[[tp^m, t(p^m - 2 - 2\frac{m}{t}), d_z/d_x]]_p$  AQECC, where  $d_z \geq d_a$ ,  $d_x \geq d_a$ , and  $d_a$  is the minimum distance of  $C^e$ . ■

### F. Code Tables

In this section we present Tables I and II containing families of AQECC available in the literature as well as the new code families constructed in this paper. In the first column we give the class and the parameters  $[[n, k, d_z/d_x]]_q$  of an AQECC; in the second column the parameter's range, and in third column, the corresponding references.

## VI. SUMMARY

We have shown how to construct new families of asymmetric stabilizer codes by applying the techniques of puncturing, extending, expanding, direct sum and the  $(u|u+v)$  construction. As examples of application of quantum code expansion, new AQECC derived from generalized Reed-Muller, quadratic residue, BCH, character and affine-invariant codes have been constructed.

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TABLE I  
FAMILIES OF AQECC

Code Family / $[[n, k, d_z/d_x]]_q$	Range of Parameters	Ref.
<b>BCH</b>		
$[[n, n - m[(\delta_1 - 1)(1 - 1/q)] - m[(\delta_2 - 1)(1 - 1/q)], d_z^*/d_x^*]]_q$	$\gcd(q, n) = 1, \text{ord}_n(q) = m,$ $q^{\lfloor m/2 \rfloor} < n \leq q^m - 1,$ $2 \leq \delta_1, \delta_2 \leq \delta_{max} = \min\{[nq^{\lfloor m/2 \rfloor}/(q^m - 1)]_j, n\},$ $\delta_1 < \delta_2^\perp \leq \delta_2 < \delta_1^\perp,$ $d_z^* = \text{wt}(C_2 \setminus C_1^\perp) \geq \delta_2,$ $d_x^* = \text{wt}(C_1 \setminus C_2^\perp) \geq \delta_1$	[1]
$[[2^m - 1, m(\delta_2 - \delta_1)/2, d_x/d_z]]_q$	$m \geq 2, 2 \leq \delta_1 < \delta_2 < \delta_{max} = 2^{\lfloor m/2 \rfloor} - 1,$ $\delta_i \equiv 1 \pmod{2}, d_x \geq \delta_1, d_z \geq \delta_{max} + 1$	[28]
$[[n, k, d_z/d_x]]_q$	$n = q^m - 1, m \geq 3$ (if $q = 3, m \geq 4$ ):	[18]
$[[n, n - m(4q - 5) - 2, d_z \geq (2q + 2)/d_x \geq 2q]]_q$		
$[[n, n - m(4q - c - 5) - 2, d_z \geq (2q + 2)/d_x \geq (2q - c)]]_q$	$0 \leq c \leq q - 2$	
$[[n, n - m(2c - l - 4) - 2, d_z \geq c/d_x \geq (c - l)]]_q$	$2 \leq c \leq q$ and $0 \leq l \leq c - 2$	
$[[n, n - m(2c - l - 6) - 2, d_z \geq c/d_x \geq (c - l)]]_q$	$q + 2 < c \leq 2q$ and $0 \leq l \leq c - q - 3$	
$[[n, n - m(4q - l - 5) - 1, d_z \geq (2q + 1)/d_x \geq (2q - l)]]_q$	$0 \leq l \leq q - 2$	
<b>Expanded BCH</b>		
$[[tn, t[n - m[(\delta_1 - 1)(1 - 1/q)] - m[(\delta_2 - 1)(1 - 1/q)], d_z^*/d_x^*]]_q$	$\gcd(q, n) = 1, \text{ord}_n(q) = m,$ $t \geq 1, q^{\lfloor m/2 \rfloor} < n \leq q^m - 1,$ $2 \leq \delta_1, \delta_2 \leq \delta_{max} = \min\{[nq^{\lfloor m/2 \rfloor}/(q^m - 1)]_j, n\},$ $\delta_1 < \delta_2^\perp \leq \delta_2 < \delta_1^\perp,$ $d_z^* = \text{wt}(C_2 \setminus C_1^\perp) \geq \delta_2,$ $d_x^* = \text{wt}(C_1 \setminus C_2^\perp) \geq \delta_1$	[1]
$[[tn, tk, d_z/d_x]]_q$	$n = q^m - 1, q = p^t, p$ odd prime, $t \geq 1,$ $m \geq 3$ (if $q = 3, m \geq 4$ ):	
$[[tn, t(n - m(4q - 5) - 2), d_z \geq (2q + 2)/d_x \geq 2q]]_p$		
$[[tn, t(n - m(4q - c - 5) - 2), d_z \geq (2q + 2)/d_x \geq (2q - c)]]_p$	$0 \leq c \leq q - 2$	
$[[tn, t(n - m(2c - l - 4) - 2), d_z \geq c/d_x \geq (c - l)]]_p$	$2 \leq c \leq q, 0 \leq l \leq c - 2$	
$[[tn, t(n - m(2c - l - 6) - 2), d_z \geq c/d_x \geq (c - l)]]_p$	$q + 2 < c \leq 2q, 0 \leq l \leq c - q - 3$	
$[[tn, t(n - m(4q - l - 5) - 1), d_z \geq (2q + 1)/d_x \geq (2q - l)]]_p$	$0 \leq l \leq q - 2$	
<b>BCH-LDPC</b>		
$[[p^{ms} - 1, k_x + k_z - p^{ms} + 1, d_z/d_x]]_p$	$\delta \leq \delta_0 = p^{\mu s} - 1$ $k_x = \dim \text{BCH}(\delta) \subseteq \mathbb{F}_p^n,$ $k_z = \dim C_{EG,c}^{(1)}(m, \mu, 0, s, p),$ $d_x \geq \delta, d_z \geq A_{EG}(m, \mu, \mu - 1, s, p)$	[28]
$[[2^{2s} - 1, 2^{2s} - 3^s - s(\delta - 1), \delta/2^s + 1]]_2$	$\delta = 2t + 1 \leq 2^s - 1$	[28]
$[[n, k_x + k_z - n, d_z/d_x]]_p$	$n = (p^{(m+1)s} - 1)/(p^s - 1)$ $\delta \leq \delta_0 = (p^{(\mu+1)s} - 1)/(p^s - 1), k_x = \dim \text{BCH}_p(\delta, n),$ $k_z = \dim C_{EG}^{(1)}(m, \mu, 0, s, p), d_x \geq \delta,$ $d_z \geq A_{EG}(m, \mu, \mu - 1, s, p)$	[28]
$[[n, n - 3^s - 3s[(\delta - 1)/2] - 1, \delta/(2^s + 2)]]_2$	$n = 2^{2s} + 2^s + 1, \delta \leq 2^{s/2} + 1$	[28]
<b>LDPC-LDPC</b>		
$[[p^{ms}, k_x + k_z - p^{ms}, d_z/d_x]]_p$	$p$ prime, $q = p^s, s \geq 1, m \geq 2,$ $1 < \mu_z < m, m - \mu_z + 1 \leq \mu_x < m,$ $k_x = \dim C_{EG}^{(1)}(m, \mu_x, 0, s, p),$ $k_z = \dim C_{EG}^{(1)}(m, \mu_z, 0, s, p),$ $d_x \geq A_{EG}(m, \mu_x, \mu_x - 1, s, p) + 1,$ $d_z \geq A_{EG}(m, \mu_z, \mu_z - 1, s, p) + 1$	[28]
<b>concatenated RS</b>		
$[[2mq, mk - 1, (\geq 2(q - k + 1))/2]]_4$	$n = 4^m, 1 \leq k \leq q$	[10]
<b>GRS</b>		
$[[mn, m(2k - n + c), d_z \geq d/d_x \geq (d - c)]]_q$	$1 < k < n < 2k + c \leq q^m,$ $k = n - d + 1, d > c + 1, c, m \geq 1$	[19]



TABLE II  
FAMILIES OF AQECC

Code Family / $[[n, k, d_z/d_x]]_q$	Range of Parameters	Ref.
RM		
$[[2^m, k, 2^{m-r_2} \geq 2^{r_1+1}]]_2$	$0 \leq r_1 < r_2 < m, k = \sum_{j=r_1+1}^{r_2} \binom{m}{j}$	[28]
Expanded GRM		
$[[lq^m, l[k(\alpha_2) - k(\alpha_1)], d_z/d_x]]_p$	$0 \leq \alpha_1 \leq \alpha_2 < m(q-1), q = p^l, p \text{ prime}, l \geq 1,$ $k(\alpha) = \sum_{i=0}^m (-1)^i \binom{m}{i} \binom{m+\alpha-iq}{\alpha-iq},$ $d_z \geq d(\alpha_2), d_x \geq d(\alpha_1^+), d(\alpha_2) = (t+1)q^u,$ $m(q-1) - \alpha_2 = (q-1)u + t, 0 \leq t < q-1,$ $d(\alpha_1^+) = (a+1)q^b, \alpha_1 + 1 = (q-1)b + a, 0 \leq a \leq q-1$	
MDS		
$[[n, n - d_1 - d_2 + 2, d_z/d_x]]_q$	$n = q - 1, d_x = d_1 < d_z = d_2$	[1]
$[[n, n - 2, 2/2]]_q$	$q \text{ prime power}, n \geq 3$	[32]
$[[n, k - 1, (n - k + 1)/2]]_q$	$q \geq n > 3, 1 < k \leq n - 2$	[32]
$[[2^m + 2, 2, 2^m/2]]_{2^m}$	$m > 0 \text{ integer}$	[32]
$[[2^m + 2, 2^m - 2, 4/2]]_{2^m}$	$m > 0, m \neq 2 \text{ integer}$	[32]
$[[n, j, d_z/d_x]]_q$	$n, k, j \in \mathbb{Z}, q \geq 5, n \leq q, 2 \leq k \leq n - 3,$ $j \leq n - k - 2, \{d_z, d_x\} = \{n - k - j + 1, k + 1\}$	[32]
$[[q + 1, 2j, d_z/d_x]]_q$	$n, k, j \in \mathbb{Z}, q \geq 5, k \geq 2, k + 2j \leq q - 1,$ $\{d_z, d_x\} = \{q - k - 2j + 2, k + 1\}$	[32]
$[[q + 1, q - 1 - 2s, (2s + 1)/3]]_q$	$q = 2^m \geq 4, s \leq q/2 - 1$	[32]
$[[2^m + 2, 2^m - 4, 4/4]]_{2^m}$	$2^m \geq 4$	[32]
$[[n, 2k - n + c, d_z \geq d/d_x \geq (d - c)]]_q$	$1 < k < n < 2k + c \leq q,$ $k = n - d + 1, d > c + 1, c \geq 1$	[19]
Expanded Character		
$[[t2^m, t[k(r_2) - k(r_1)], d_z/d_x]]_p$	$q = p^t, p \text{ odd prime}, t \geq 1,$ $k(r) = \sum_{i=0}^r \binom{m}{i}, d_z \geq 2^{m-r_2}, d_x \geq 2^{r_1+1}$	
QR		
$[[p, 1, d_z/d_x]]_q$	$p \text{ prime}, p \equiv 1 \pmod{4},$ $q = p_1^t, p \nmid p_1, q \text{ is a quadratic residue mod } p,$ $d_z \geq \sqrt{p}, d_x \geq \sqrt{p}$	[16]
$[[p, 1, d_z/d_x]]_q$	$p \text{ prime}, p \equiv 3 \pmod{4},$ $q = p_1^t, p \nmid p_1, q \text{ is a quadratic residue mod } p,$ $d_z \geq d, d_x \geq d, d^2 - d + 1 \geq p$	[16]
Expanded QR		
$[[tp, t, d_z/d_x]]_{p_*}$	$p \text{ prime}, p \equiv 1 \pmod{4}, q = p_*^t,$ $t \geq 1, p \nmid p_*, q \text{ is a quadratic residue mod } p,$ $d_z \geq \sqrt{p}, d_x \geq \sqrt{p}$	
$[[tp, t, d_z/d_x]]_{p_*}$	$p \text{ prime}, p \equiv 3 \pmod{4}, q = p_*^t,$ $t \geq 1, p \nmid p_*, q \text{ is a quadratic residue mod } p,$ $d_z \geq d, d_x \geq d, d^2 - d + 1 \geq p$	
Affine-Invariant		
$[[tp^m, t(p^m - 2 - 2\frac{m}{t}), d_z/d_x]]_p$	$q = p^t,$ $p > 3, m > 2, d_z \geq d_a, d_x \geq d_a,$ $d_a \text{ is given in Theorem 5.11}$	
Product code		
$[[((q-1)^2, (q-d_1)(q-d_3) - (q-d_2)(q-d_4), d_z/d_x)]]_q$	$2 \leq d_1 \leq d_2 < q-1, 2 \leq d_3 \leq d_4 < q-1,$ $d_z \geq \max\{d_1 d_3, \min\{q - d_2, q - d_4\}\},$ $d_x \geq \min\{d_1 d_3, \min\{q - d_2, q - d_4\}\}$	[21]